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Pseudovarieties, Generalized Varieties and Similarly Described Classes

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INTRODUCTION

We consider the notion of a *pseudovariety*, introduced by Eilenberg and Schützenberger in [2], for monoids, as a class of *finite* algebras closed under the formation of homomorphic images, subalgebras and *finite* direct products. First we devise a related notion and define a *generalized variety* to be a class of algebras (not necessarily finite) closed under formation of homomorphic images, subalgebras, finite direct products and arbitrary direct powers. We show that a class is a pseudovariety if and only if it consists of the finite members of some generalized variety.

We establish several equivalent characterizations of generalized varieties from which we can deduce those for pseudovarieties and show how the characterization of pseudovarieties in terms of sequences of identities, obtained in [2] for monoids, applies whenever the number of operations is finite. When this is not the case, examples are given to show that the conditions involved are not equivalent. In Section 4 we note that generalized varieties which are not varieties are not elementary classes. Section 5 discusses a slightly different application of sequences of identities.

In Section 6, we show how results analogous to those of Sections 1 and 2 can be obtained from known sets of sentences preserved under various operations. We can therefore characterize those classes of finite algebras closed under each combination of formation of homomorphic images, subalgebras, finite direct products and finite direct powers. For example, for algebras of finite type, a class of finite algebras is closed under formation of subalgebras and finite direct powers if and only if, for some sequence of implications, the class consists of exactly those finite algebras which satisfy all the implications in the sequence from some point on. A table of these results is given in Section 7.

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Preliminaries

All algebras under consideration are assumed to have the same similarity type. The set of all identities of this type is denoted by E. Where relevant, classes of algebras are assumed closed under isomorphism.

For each algebra A, $\mathrm{Id}(A)$ denotes the set of identities true in A. For a class K, $\mathrm{Id}(K)$ denotes $\bigcap \{\mathrm{Id}(A) | A \in K\}$. A *variety* is a class K of algebras for which there exists $E_0 \subseteq E$ with $K = \{A \mid \mathrm{Id}(A) \supseteq E_0\}$.

For a class K of algebras, $H(K)$, $S(K)$, $P(K)$, $P_f(K)$, $\mathrm{Pow}(K)$ and $\mathrm{Pow}_f(K)$ denote, respectively, the classes of homomorphic images, subalgebras, direct products, finite direct products, direct powers and finite direct powers of members of K. Also $P_r(K)$, $\mathrm{Pow}_r(K)$, $P_u(K)$, $\mathrm{Pow}_u(K)$ and $E(K)$ denote, respectively, the classes of reduced products, reduced powers, ultraproducts, ultrapowers and elementary subalgebras of members of K, as explained below.

By Birkhoff's Theorem (see [3]) K is a variety iff $K = HSP(K)$. Every class K is contained in a smallest variety $HSP(K)$ also denoted by $\mathrm{Var}(K)$. A family Γ of sets or of classes is said to be *directed* if for all $A, B \in \Gamma$ there exists $C \in \Gamma$ with $A \subseteq C$ and $B \subseteq C$.

A *filter* over a set I is a family of subsets of I closed under formation of supersets and finite intersections. A filter is *proper* if it does not contain the empty set. An *ultrafilter* is a maximal proper filter. A family of sets is said to have the *finite intersection property* if the intersection of any finite subfamily is non-empty. A family with this property can be extended to a proper filter and so, by Zorn's Lemma, to an ultrafilter.

A *reduced product* of algebras $\{A_i | i \in I\}$ is a quotient of $\prod_{i \in I} A_i$ determined, for some filter F over I, by the congruence $\xi \equiv \eta$ iff $\{i \in I \mid \xi(i) = \eta(i)\} \in F$. In the case where F is an ultrafilter, this is called an *ultraproduct*. In the case where the algebras A_i are all the same, these quotients are called *reduced powers* and *ultrapowers*.

We say that A is an *elementary subalgebra* of B if it is a subalgebra in such a way that each relation on the set A defined by a first-order formula is the restriction of the relation defined in the same way on B. Using a result of Shelah [4], an equivalent condition is that there exists an isomorphism between ultrapowers of A and B which commutes with the diagonal embeddings.

These notions are relevant because of the result (see [1]) that a class K is an elementary class if and only if K is closed under the formation of ultraproducts and elementary subalgebras. Using Shelah's result, this condition may be replaced by the condition that K is closed under the formation of ultraproducts and the extraction of "ultraroots," that is, if some ultrapower of A is in K, then A is in K.

1. Generalized Varieties

We define a *generalized variety* to be a class of algebras satisfying any one of the conditions of the following theorem.

THEOREM 1. *The following are equivalent conditions on a class K of algebras.*

(1) *K is closed under H, S, P_f and* Pow.

(2) $K = HSP_f\mathrm{Pow}(K)$.

(3) *K is the union of some directed family of varieties.*

(4) *There exists a filter F over E such that, for all algebras A,*

$$A \in K \Leftrightarrow \mathrm{Id}(A) \in F.$$

Proof. $(1) \Rightarrow (2)$ is immediate.

$(2) \Rightarrow (3)$ For each finite $K_0 \subseteq K$, $\mathrm{Var}(K_0) = HSP(K_0) = HSP_f\mathrm{Pow}(K_0) \subseteq K$. Also $\mathrm{Var}(K_1) \cup \mathrm{Var}(K_2) \subseteq \mathrm{Var}(K_1 \cup K_2)$. Hence K is the union of the family $\{\mathrm{Var}(K_0) \mid K_0$ is a finite subset of $K\}$ of varieties, which is directed under $\subseteq$.

$(3) \Rightarrow (4)$ Suppose that K is the union of a family Γ of varieties directed under $\subseteq$. Define $F = \{S \subseteq E \mid$ for some $V \in \Gamma$, $\mathrm{Id}(V) \subseteq S\}$. Then

$$\begin{aligned} \mathrm{Id}(A) \in F &\Leftrightarrow \text{for some} \quad V \in \Gamma,\ \mathrm{Id}(A) \supseteq \mathrm{Id}(V) \\ &\Leftrightarrow \text{for some} \quad V \in \Gamma, A \in V \\ &\Leftrightarrow A \in K. \end{aligned}$$

$(4) \Rightarrow (1)$ Immediate from $\mathrm{Id}(H(A)) \supseteq \mathrm{Id}(A)$, $\mathrm{Id}(S(A)) \supseteq \mathrm{Id}(A)$, $\mathrm{Id}(\mathrm{Pow}(A)) \supseteq \mathrm{Id}(A)$ and $\mathrm{Id}(A_1 \times A_2 \times \cdots \times A_n) = \mathrm{Id}(A_1) \cap \mathrm{Id}(A_2) \cap \cdots \cap \mathrm{Id}(A_n)$. ∎

Comments. We could restate condition (3) or (4) as: K is the union of an ideal in the lattice of varieties.

Theorem 1 may be viewed as a generalization of Birkhoff's Theorem characterizing varieties as classes K with $K = HSP(K)$. If K is a generalized variety determined by a filter F over E, then K is also closed under formation of arbitrary direct products if and only if F is closed under arbitrary intersections. This is equivalent to the condition that F is principal, that is, $F = \{S \subseteq E \mid E_0 \subseteq S\}$ for some $E_0 \subseteq E$, or equivalently, K is the variety determined by E_0.

Using condition (2) of Theorem 1, it is clear that every class K is included in a smallest generalized variety, namely, $\mathrm{Gen}(K) = HSP_f\mathrm{Pow}(K)$. We then have

$$A \in \mathrm{Gen}(K) \Leftrightarrow \mathrm{Id}(A) \in F$$

where $F = \{S \subseteq E \mid \mathrm{Id}(K_0) \subseteq S \text{ for some finite } K_0 \subseteq K\}$.

2. Pseudovarieties

Let us define a *pseudovariety* to be any class of finite algebras closed under the formation of homomorphic images, subalgebras and finite direct products. Pseudovarieties are related to generalized varieties by the following theorem.

THEOREM 2. *A class of algebras is a pseudovariety if and only if it consists of the finite members of some generalized variety.*

Proof. If K is a generalized variety then its finite members certainly form a pseudovariety, since H, S and P_f do not lead out of the class of finite algebras.

Conversely, if K_0 is a pseudovariety and $K = \mathrm{Gen}(K_0)$, we claim that every finite member of K is a member of K_0. For, suppose that $A \in K$ and A is finite. Then there exist $A_1, A_2,..., A_n \in K_0$, sets $I_1, I_2,..., I_n$, an algebra $B \subseteq A_1^{I_1} \times \cdots \times A_n^{I_n}$ and a surjective homomorphism $\phi: B \to A$. For each $a \in A$, we may choose $b_a = (\xi_{1a},..., \xi_{na}) \in B$ with $\phi(b_a) = a$. This choice gives equivalence relations $\equiv_k$ on each I_k, each having only finitely many equivalence classes, defined by $i \equiv_k j$ if for all $a \in A$, $\xi_{ka}(i) = \xi_{ka}(j)$. But now, letting $J_k = I_k/\equiv_k$, we have a homomorphism from $A_1^{I_1} \times \cdots \times A_n^{I_n}$ to $A_1^{J_1} \times \cdots \times A_n^{J_n}$ which is one–one on the subalgebra of B generated by the b_a. Thus, since each J_k is finite, we have $A \in HSP_f(K_0)$ and so $A \in K_0$. ∎

In [2] it was shown that each pseudovariety of monoids is characterized by a *sequence* of identities. The next theorem generalizes this.

THEOREM 3. *Let C be a countable class of algebras. The following conditions on $K_0 \subseteq C$ are equivalent.*

(1) There is a generalized variety K with $K_0 = K \cap C$.

(2) There is a chain $V_1 \subseteq V_2 \subseteq \cdots$ of varieties with $K_0 = (\bigcup V_n) \cap C$.

(3) *There is a sequence $e_1, e_2,...$ of identities such that for $A \in C$,*

$$A \in K_0 \Leftrightarrow e_n \in \mathrm{Id}(A) \qquad \textit{for all but finitely many } n.$$

Proof. (3) ⇒ (2) If $e_1, e_2, \ldots$ is such a sequence of identities, let V_n be the variety of all algebras satisfying each of the identities $e_n, e_{n+1}, \ldots$.

(2) ⇒ (1) Immediate using Theorem 1.

(1) ⇒ (3) Suppose K is a generalized variety with $K_0 = K \cap C$. Let $K_0 = \{A_1, A_2, \ldots\}$ and $C - K_0 = \{B_1, B_2, \ldots\}$. For each j, $B_j \notin K$, hence for each $i. j$, $B_j \notin \operatorname{Var}\{A_1, A_2, \ldots, A_i\}$. We may thus choose an identity $e_{ij} \in \operatorname{Id}\{A_1, \ldots, A_i\} - \operatorname{Id}(B_j)$. Let $e_1, e_2, \ldots$ be any listing of those e_{ij} with $i > j$. Then for each j, e_n is of the form e_{ij} for infinitely many n and each $e_{ij} \notin \operatorname{Id}(B_j)$, while for each k, e_n is e_{ij} with $i \geqslant k$ except for finitely many n, and for all $i \geqslant k$, $e_{ij} \in \operatorname{Id}(A_k)$. ∎

Comment. It is clear from the proof that this theorem also holds for any class for which $\{\operatorname{Id}(A) \mid A \in C\}$ is countable.

COROLLARY. *If C is the class of all finite algebras of some finite type then conditions* (1), (2) *and* (3) *above are also equivalent to*:

(4) $K_0 = HSP_f(K_0)$.

Proof. (1) ⇔ (4) is immediate from Theorem 2.

3. EXAMPLES

In the case where C is the class of all finite algebras of some type, not necessarily finite, the conditions of Theorem 3 and its Corollary on $K_0 \subseteq C$ are still related by the implications (3) ⇒ (2) ⇒ (1) ⇔ (4). We show that no other implications hold.

EXAMPLE 1. Let $\{c_n \mid n \in \mathbb{N}\} \cup \{d\}$ be nullary operation symbols. Let F be a non-principal ultrafilter over $\mathbb{N}$. Let K_0 be the class of all the finite algebras of this type for which $\{k \mid c_k = d\} \in F$.

By Theorem 2, K_0 is a pseudovariety. Let $V_1 \subseteq V_2 \subseteq \cdots$ be varieties. We show that K_0 does not consist of exactly the finite members of $\bigcup V_n$. Let $S_n = \{i \mid (c_i = d) \in \operatorname{Id}(V_n)\}$. Then $S_1 \supseteq S_2 \supseteq \cdots$.

If any $S_n \notin F$, then the free algebra, B, on 0 generators in V_n has $\{i \mid (c_i = d) \in \operatorname{Id}(B)\} = S_n$ and we may take a 2-element quotient A of B for which $\{i \mid (c_i = d) \in \operatorname{Id}(A)\} = S_n$. Then A is finite, $A \in V_n$ but $A \notin K_0$.

We may thus suppose that each $S_n \in F$. Since F is non-principal, no S_n is a singleton. There is then a partition of $\mathbb{N}$ into two components A and B such that for no n do we have $S_n \subseteq A$ or $S_n \subseteq B$. For, either there is a least S_n in which case let A and B both intersect S_n, or infinitely many of the disjoint sets $S_n - S_{n+1}$ are non-empty, in which case let A and B each intersect infinitely many sets $S_n - S_{n+1}$.

Since F is an ultrafilter, either $A \in F$ or $B \in F$. Thus there exists $A \in F$ such that for no n is $S_n \subseteq A$. Let A be the 2-element algebra for which $\{i \mid (c_i = d) \in \mathrm{Id}(A)\} = A$. Then $A \in K_0$ but $A \notin \bigcup V_n$.

Comment. A similar argument shows that in this example K_0 is not the class of finite members of the union of any family of varieties linearly ordered by inclusion.

EXAMPLE 2. Let $\{c_{im} \mid i, m \in \mathbb{N}\}$ be nullary operations. Let V_n be the variety of algebras of this type determined by the identities $\{c_{im} = c_{jm} \mid m \geqslant n\}$. Let K_0 be the finite members of $\bigcup V_n$. Thus condition (2) holds for K_0. We show that condition (3) fails.

Suppose that $e_0, e_1, \ldots$ is a sequence of identities.

Let U_n be the variety determined by $e_n, e_{n+1}, \ldots$.

We consider two cases. First, suppose that there exists m such that for all n there exist $i \neq j$ with $(c_{im} = c_{jm}) \in \mathrm{Id}(U_n)$. We may then find a partition of $\mathbb{N}$ into two components S and T so that for all n there exist $i \in S$ and $j \in T$ with $(c_{im} = c_{jm}) \in \mathrm{Id}(U_n)$. There is thus a 2-element algebra $A \in V_{m+1}$ for which $e_k \notin \mathrm{Id}(A)$ for infinitely many k.

Second, suppose that this is not the case. We may consider the completely free algebra F on 0 generators as consisting of disjoint sets $C_m = \{c_{im}\}$, and the free algebra in U_0 as the quotient of F by an equivalence relation $\equiv$. By assumption, for each m there are only finitely many i such that $c_{im} \equiv c_{jn}$ for some $c_{jn} \neq c_{im}$. We may thus partition the set of $\equiv$ classes into two components so that for each m, each component contains a class intersecting C_m. Thus there is a 2-element algebra $A \in U_0$ with $A \notin \bigcup V_n$. So in each case we do not have that for all finite A, $A \in \bigcup V_n \Leftrightarrow A \in \bigcup U_n$.

4. ELEMENTARY CLASSES

Unlike many other classes characterized by theorems such as Theorem 1, generalized varieties are not always elementary classes. In fact, we have the following.

THEOREM 4. *The following conditions are equivalent on a generalized variety K.*

(1) *K is an elementary class.*

(2) *K is closed under formation of ultraproducts.*

(3) *K is a variety.*

Proof. $(3) \Rightarrow (1)$ since an equational class is always an elementary class.

$(1) \Rightarrow (2)$ by the known characterization of elementary classes.

$(2) \Rightarrow (3)$ Assume that K is a generalized variety closed under the formation of ultraproducts.

Let E be the set of all identites. Let $\{s_i \mid i \in I\}$ be the family of finite subsets of $E - \mathrm{Id}(K)$. For each $e \in E - \mathrm{Id}(K)$ we may choose $A_e \in K$ with $e \notin \mathrm{Id}(A_e)$. For each $i \in I$ let $A_i = \prod_{e \in s_i} A_e$. For each $k \in I$, let $J_k = \{i \in I \mid s_i \supseteq s_k\}$. Then the family $\{J_k \mid k \in I\}$ has the finite intersection property and so there is an ultrafilter U over I containing each J_k. Let B be the ultraproduct

$$\frac{\prod_{i \in I} A_i}{U}.$$

Let $e \notin \mathrm{Id}(K)$. Then, letting $\{e\} = s_k$, we have $e \notin \mathrm{Id}(A_i)$ for all $i \in J_k$. So $\{i \mid e \notin \mathrm{Id}(A_i)\} \in U$, hence $e \notin \mathrm{Id}(B)$. Thus $\mathrm{Id}(K) \supseteq \mathrm{Id}(B)$ and so $\mathrm{Var}(K) \subseteq \mathrm{Var}(B)$. But, by assumption, $B \in K$ and so $\mathrm{Var}(B) \subseteq K$. Thus $\mathrm{Var}(K) \subseteq K$ and so K is a variety. ∎

5. Finitely Generated Algebras

A slightly different application of sequences of identities is the following.

Theorem 5. *For each variety V of algebras of finite type, there exists a sequence $e_1, e_2, \ldots$ of identities such that, for each* finitely generated *algebra $A \in V$,*

$$A \text{ is finite} \Leftrightarrow e_n \in \mathrm{Id}(A) \qquad \text{for all sufficiently large } n.$$

Proof. Let $A_1, A_2, \ldots$ be the finite members of V. For each n, let $e_{n1}, \ldots, e_{nk_n}$ be a basis for the n-ary identities of $\mathrm{Var}(A_1, \ldots, A_n)$. Then the sequence

$$e_{11}, \ldots, e_{1k_1}, e_{21}, \ldots, e_{2k_2}, e_{31}, \ldots$$

has the desired property. ∎

This result is different in spirit from that characterizing pseudovarieties. There we were concerned with distinguishing the members of a class from among the finite algebras. Here we wish to distinguish the finite algebras from among all the (finitely generated) members of a class.

However, the result can be viewed in terms of the previous notions. Let K be the smallest generalized variety containing all the finite algebras of V. We do not know when K can be determined by a sequence of identities, but Theorem 5 shows that there is a generalized variety K' which *is* so determined and for which $K \subseteq K' \subseteq L$, where L is the class of locally finite members of V.

Comments. In the case where V is the variety of groups, the generalized variety K itself is determined by a sequence of identities. This follows from the result that the identities of a finite group have a finite basis. In general, the class K' presumably has otherwise no special significance, since its definition depends on the particular enumeration of the finite members of V.

Returning to the result of Theorem 5, in the case where V is the variety of abelian groups, one such sequence is $x = 1, x^2 = 1, \ldots, x^{n!} = 1, \ldots$. For other varieties, such as those of groups, semigroups or lattices, we know of no simply described sequence with the desired effect.

6. Pseudo-$\mathscr{O}$-Classes

The characterization of pseudovarieties in terms of sequences of identities rests on that of varieties in terms of identities. Similarly, other characterizations of classes closed under certain operations yield corresponding results about classes of finite algebras. We give some general results and a list of applications.

$\mathscr{O}$-Classes

Let $\mathscr{O}$ be a family of operations on classes of algebras. For each class K, let $\mathscr{O}(K)$ be the smallest class K' such that $K \subseteq K'$ and such that for each $0 \in \mathscr{O}$ and each $K_1 \subseteq K'$, we have $0(K_1) \subseteq K'$.

Let us say that K is an *$\mathscr{O}$-class* if $K = \mathscr{O}(K)$ and that K is a *generalized $\mathscr{O}$-class* if K is a directed union of $\mathscr{O}$-classes. Let Fin denote the class of finite algebras and let us say that K_0 is a *pseudo-$\mathscr{O}$-class* if $K_0 = K \cap \mathrm{Fin}$ for some generalized $\mathscr{O}$-class K.

For any set $\mathscr{S}$ of sentences of any logic, let $\mathscr{S}(A)$ denote the set of sentences of $\mathscr{S}$ true in the algebra A and for a class K of algebras let $\mathscr{S}(K) = \bigcap \{\mathscr{S}(A) \mid A \in K\}$.

Now assume that the set $\mathscr{S}$ *characterizes* the operation $\mathscr{O}$ in the sense that, for each algebra A and class of algebras K,

$$A \in \mathscr{O}(K) \Leftrightarrow \mathscr{S}(A) \supseteq \mathscr{S}(K).$$

An immediate consequence of this assumption is the following.

THEOREM 6. (i) *$K = \mathscr{O}(K)$ iff there exists a set of sentences $\mathscr{S}_0 \subseteq \mathscr{S}$ such that*

$$A \in K \Leftrightarrow \mathscr{S}(A) \supseteq \mathscr{S}_0.$$

[*That is, the* $\mathcal{O}$*-classes are exactly those classes axiomatized by some set of sentences of* $\mathcal{S}$.]

(ii) $\mathcal{S}(K) = \mathcal{S}(\mathcal{O}(K))$.

[*That is, the sentences of* $\mathcal{S}$ *are "preserved" under the operations of* $\mathcal{O}$.]

Proof. (i) If $K = \mathcal{O}(K)$ then, by assumption, $\mathcal{O}(K) = \{A \mid \mathcal{S}(A) \supseteq \mathcal{S}(K)\}$, so take $\mathcal{S}_0 = \mathcal{S}(K)$.

Conversely, if $A \in K \Leftrightarrow \mathcal{S}(A) \supseteq \mathcal{S}_0$, then $\mathcal{S}(K) \supseteq \mathcal{S}_0$. So $A \in \mathcal{O}(K) \Rightarrow \mathcal{S}(A) \supseteq \mathcal{S}(K) \Rightarrow \mathcal{S}(A) \supseteq \mathcal{S}_0 \Rightarrow A \in K$. Thus $\mathcal{O}(K) \subseteq K$, so $\mathcal{O}(K) = K$.

(ii) Since $K \subseteq \mathcal{O}(K)$, certainly $\mathcal{S}(\mathcal{O}(K)) \subseteq \mathcal{S}(K)$. For $A \in \mathcal{O}(K)$, we have $\mathcal{S}(A) \supseteq \mathcal{S}(K)$. So $\mathcal{S}(\mathcal{O}(K)) = \bigcap \{\mathcal{S}(A) \mid A \in \mathcal{O}(K)\} \supseteq \mathcal{S}(K)$. ∎

In this situation, close analogues of Theorem 1, Theorem 3 and its Corollary may now be proved. Thus, the generalized $\mathcal{O}$-classes may be characterized as follows.

THEOREM 7. *The following are equivalent for a class* K *of algebras.*

(1) *K is the union of a directed family of* $\mathcal{O}$*-classes.*

(2) $K = \bigcup \{\mathcal{O}(K_0) \mid K_0 \subseteq K$ *and* K_0 *finite*$\}$.

(3) *There is a filter* F *over* $\mathcal{S}$ *for which* $A \in K \Leftrightarrow \mathcal{S}(A) \in F$.

Proof. (2) $\Rightarrow$ (1) Since $\mathcal{O}(K_1) \cup \mathcal{O}(K_2) \subseteq \mathcal{O}(K_1 \cup K_2)$.

(1) $\Rightarrow$ (3) Let K be the union of a directed family $\{K_i \mid i \in I\}$.

Let F be the filter over I generated by the $\mathcal{S}(K_i)$. Then $A \in K \Rightarrow A \in$ some $K_i \Rightarrow \mathcal{S}(A) \supseteq \mathcal{S}(K_i) \Rightarrow \mathcal{S}(A) \in F$. Conversely, if $\mathcal{S}(A) \in F$, then $\mathcal{S}(A) \supseteq \mathcal{S}(K_{i_1}) \cap \cdots \cap \mathcal{S}(K_{i_n})$ for some $i_1, \ldots, i_n \in I$. So $\mathcal{S}(A) \supseteq$ some $\mathcal{S}(K_i)$, since the $\{K_i\}$ are directed. Thus $A \in K_i$ and so $A \in K$.

(3) $\Rightarrow$ (2) If the filter F has the property described and if $K_0 = \{B_1, \ldots, B_n\} \subseteq K$, and $A \in \mathcal{O}(K_0)$, then $\mathcal{S}(A) \supseteq \mathcal{S}(K_0) = \mathcal{S}(B_1) \cap \cdots \cap \cap \mathcal{S}(B_n)$, so $\mathcal{S}(A) \in F$ and so $A \in K$. ∎

Now let the single operation $\mathcal{O}'$ on classes of algebras be defined by

$$\mathcal{O}'(K) = \bigcup \{\mathcal{O}(K_1) \cap \mathrm{Fin} \mid K_1 \subseteq K \text{ and } K_1 \text{ is finite}\}.$$

The pseudo-$\mathcal{O}$-classes may be characterized as follows.

THEOREM 8. *The following are equivalent for a class* K_0 *of finite algebras.*

(1) $K_0 = K \cap \mathrm{Fin}$ *for some generalized* $\mathcal{O}$*-class* K.

(2) $K_0 = \mathcal{O}'(K_0)$.

(3) *There is a filter* F *over* $\mathscr{S}$ *such that* $A \in K_0 \Leftrightarrow A \in \mathrm{Fin}$ *and* $\mathscr{S}(A) \in F$.

Proof. (1) $\Rightarrow$ (2) Let $K_0 = K \cap \mathrm{Fin}$. If $A \in \mathscr{O}'(K_0)$, then $A \in \mathscr{O}(K_1) \cap \mathrm{Fin}$ for some $K_1 \subseteq K_0$ with K_1 finite. Thus $K_1 \subseteq K$ and so $\mathscr{O}(K_1) \subseteq K$. So $A \in K$ and since $A \in \mathrm{Fin}$, $A \in K \cap \mathrm{Fin} = K_0$.

(2) $\Rightarrow$ (1) Suppose that $K_0 = \mathscr{O}'(K_0)$. Let $K = \bigcup \{\mathscr{O}(K_1) \mid K_1 \subseteq K_0$ and K_1 is finite$\}$. Then $K_0 \subseteq K \cap \mathrm{Fin}$. If $A \in K \cap \mathrm{Fin}$, then $A \in \mathscr{O}'(K_0)$. So $K \cap \mathrm{Fin} \subseteq K_0$.

(1) $\Rightarrow$ (3) Immediate from Theorem 7.

THEOREM 9. *For algebras of finite type, the conditions of Theorem* 8 *are also equivalent to*:

(4) *There is a sequence* $\phi_1, \phi_2, \ldots$ *of sentences of* $\mathscr{S}$ *such that*

$$A \in K_0 \Leftrightarrow A \in \mathrm{Fin} \quad \textit{and} \quad \phi_n \in \mathscr{S}(A) \quad \textit{for all sufficiently large } n.$$

Proof. (4) $\Rightarrow$ (1) Let K_n be the $\mathscr{O}$-class axiomatized by $\{\phi_n, \phi_{n+1}, \ldots\}$ and let $K = \bigcup K_n$. Then $K_0 = K \cup \mathrm{Fin}$.

(2) $\Rightarrow$ (4) Let $K_0 = \{A_1, A_2, \ldots\}$ and let $\mathrm{Fin} - K_0 = \{B_1, B_2, \ldots\}$. Since $B_j \notin K_0$, for each i, j, we have $B_j \notin \mathscr{O}(A_1, \ldots, A_i)$. So we may choose $\psi_{ij} \in (\mathscr{S}(A_1) \cap \cdots \cap \mathscr{S}(A_i)) - \mathscr{S}(B_j)$. Now let $\{\phi_n\}$ enumerate those ψ_{ij} with $i > j$. ∎

7. SOME APPLICATIONS

Theorems 8 and 9 show that, for certain families $\mathscr{O}'$ of operations and for certain sets $\mathscr{S}$ of sentences, the classes of finite algebras closed under $\mathscr{O}'$ are described by filters over $\mathscr{S}$ or, in the case of algebras of finite type, by sequences from $\mathscr{S}$.

For example, we may take the known result for quasivarieties, that $\mathscr{O} = \{S, P, P_u\}$ [or equivalently $\{S, P_r\}$] is characterized by the set $\mathscr{S}$ of implications. To simplify calculations, we may use that, for every class K, $PS(K) \subseteq SP(K)$ and likewise that $P_u S \subseteq SP_u$ and $P_u P \subseteq P_r \subseteq SPP_u$ to see that $\mathscr{O}(K) = K$ iff $K = SPP_u(K)$.

Now, since $SPP_u = SP_r$ and since, for each finite K_1, $P_r(K_1) = P_f\mathrm{Pow}_r(K_1)$, we see that $\mathscr{O}'(K) = K$ iff $K = SP_f\mathrm{Pow}_r(K)$. Thus, by Theorem 7 we first obtain the following equivalent characterizations of *generalized quasivarieties.*

THEOREM 10. *The following are equivalent on a class K closed under isomorphism.*

(1) *K is the union of a directed family of quasivarieties.*

(2) *K is closed under S, P_f, Pow_r.*

(3) $K = SP_f\mathrm{Pow}_r(K)$.

(4) *There is a filter F over the set $\mathscr{S}$ of implications for which $A \in K \Leftrightarrow \mathscr{S}(A) \in F$.* ∎

Now let K be a class of *finite* algebras. From the definition, K is a *pseudo-quasivariety* iff whenever $A \in SP_f\mathrm{Pow}_r(K)$ and A is finite, then $A \in K$. But this can be simplified, much as in Theorem 2, since for such an A, $A \in SP_f(K)$. Thus from Theorems 8 and 9 we obtain a characterization of those classes of finite algebras closed under S and P_f.

THEOREM 11. *The following are equivalent on a class K of finite algebras.*

(1) *K consists of the finite members of the union of some directed family of quasivarieties.*

(2) *K is closed under S, P_f.*

(3) $K = SP_f(K)$.

(4) *There is a filter F over the set $\mathscr{S}$ of all implications for which $A \in K \Leftrightarrow A$ is finite and $\mathscr{S}(A) \in F$.*

And, for algebras of finite type,

(5) *There is a sequence $i_1, i_2,\ldots$ of implications for which $A \in K \Leftrightarrow A$ is finite and i_n is true in A for all but finitely many n.* ∎

Here we have found operations $\mathscr{O}'' = \{S, P_f\}$ such that K is a pseudo-$\mathscr{O}$-class iff K is an $\mathscr{O}''$-class of finite algebras. In fact, each subset of the operations $H, S, P_f, \mathrm{Pow}_f$ arises in this way, by taking $\mathscr{O}$ to be a suitable

TABLE I

$\mathscr{O}$	$\mathscr{O}'$	$\mathscr{O}''$	$\mathscr{S}$
SP_u	$S\mathrm{Pow}_u$	S	universal (sentences)
EHP_u	$EH\mathrm{Pow}_u$	H	positive
HSP_u	$HS\mathrm{Pow}_u$	HS	universal positive
EP_r	$EP_f\mathrm{Pow}_r$	P_f	Horn
$E\mathrm{Pow}_rP_u$	$E\mathrm{Pow}_r$	Pow_f	disjunctions of Horn
SPP_u	$SP_f\mathrm{Pow}_r$	SP_f	implications
$SP_u\mathrm{Pow}$	$S\mathrm{Pow}_r$	$S\mathrm{Pow}_f$	disjunctions of implications
EHP	$EHP_f\mathrm{Pow}$	HP_f	positive Horn
$EHP_u\mathrm{Pow}$	$EH\mathrm{Pow}$	$H\mathrm{Pow}_f$	disjunctions of positive Horn
HSP	$HSP_f\mathrm{Pow}$	HSP_f	identities
$HSP_u\mathrm{Pow}$	$HS\mathrm{Pow}$	$HS\mathrm{Pow}_f$	disjunctions of identities

choice from H, S, P, P_r, Pow, Pow_r plus, in each case, E and P_u unless they are redundant. For each of these combinations, we have a known set of sentences characterizing $\mathcal{O}$ and so obtain, by Theorems 7, 8 and 9, corresponding characterizations of the $\mathcal{O}'$-classes of algebras and of the $\mathcal{O}''$-classes of finite algebras, as in Theorems 10 an 11.

These results are summarized in Table 1. The sorts of sentences mentioned are as follows.

Descriptions of sentences. We need consider only sentences of the general form

$$Q_1 x_1 \cdots Q_n x_n (\phi_1(\bar{x}) \mathbin{\&} \cdots \mathbin{\&} \phi_m(\bar{x}))$$

where each Q_i is $\forall$ or $\exists$, each $\phi_i(\bar{x})$ is $\psi_{i1}(\bar{x}) \wedge \cdots \wedge \psi_{ik_i}(\bar{x})$, each $\psi_{ij}(\bar{x})$ is either $p(\bar{x}) = q(\bar{x})$ [unnegated] or $\neg(p(\bar{x}) = q(\bar{x}))$ [negated] and p, q are polynomial symbols.

Here $\bar{x}$ abbreviates the sequence of variables $x_1, x_2, \dots, x_n$ and $\forall, \exists, \&, \wedge, \neg$ denote, respectively, for all, there exists, and, or, not.

A sentence of this form is then:

an *identity* if each Q_i is $\forall$, $m = 1$, $k_1 = 1$ and ψ_{11} is unnegated,
positive if each ψ_{ij} is unnegated,
universal if each Q_i is $\forall$,
positive universal if it is both positive and universal [without loss of generality, $m = 1$],
a *Horn* sentence if, for each i, at most one ψ_{ij} is unnegated,
an *implication* if it is a universal Horn sentence and $m = 1$,
a *positive Horn* sentence if each $k_i = 1$ and each ψ_{i1} is unnegated.

The *disjunction* of sentences $\theta_1, \theta_2, \dots, \theta_p$ is the sentence $\theta_1 \vee \theta_2 \vee \cdots \vee \theta_p$.

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